A Jacobian criterion of smoothness for algebraic diamonds

Laurent Fargues (CNRS/IMJ) - Joint with P. Scholze

Motivation

- *Problem* : Give a meaning to Bun_G is "smooth of dimension 0", $U \to \mathrm{Bun}_G$ is a "smooth chart" on Bun_G.
  - $\mathrm{Bun}_G = v$-stack of G−bundles / curve (stack on $\mathrm{Perf}_{\mathbb{F}_p}$)
  - $U =$ locally spatial diamond

  $\rightsquigarrow$ no infinitesimal crietrion (perfect world), no Jacobian criterion
- *Solution* : cohomological smoothness ("Étale cohomology of diamonds")
  $\rightsquigarrow$ morphisms that satisfy relative Poincaré duality

Cohomological smoothness

- *Spaces* : $\mathfrak{X}$ = small v-stack
  - small= $\exists v$-presentation by perfectoid spaces
  - $v-$stack=Stack on $\mathrm{Perf}_{\mathbb{F}_p}$ + v-topology
  - *Example* : Beauville-Laszlo : $\mathrm{Gr}_G^{B_{dR}} \longrightarrow \mathrm{Bun}_G$
    $\mathrm{Gr}_G^{B_{dR}} = B_{dR}$-affine Grassmanian = ind-diamond
    v-surjective (analog of Drinfeld-Simpson known)
- *Coefficients* : $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $(n,p)=1$

$$D_{\text{ét}}(\mathfrak{X},\Lambda) = \Big\{\mathscr{F} \in \underbrace{D(\mathfrak{X},\Lambda)}_{\substack{\text{cartesian}\\ v-\text{sheaves}}} \mid \forall \underbrace{U}_{\substack{\text{strictly tot disc}\\ \text{perf space}}} \xrightarrow{f} \mathfrak{X},\ f^*\mathscr{F} \in \underbrace{D(U_{\text{ét}},\Lambda)}_{=D(|U|,\Lambda)}\Big\}$$
$$= D_{cart}(|U_\bullet|,\Lambda)$$

for some simplicial strictly tot. disc. perf. space $U_\bullet$

- *Morphisms* : $\mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ representable in loc. spatial diamonds (compactifiable, dim trg f< $+\infty$)

Cohomological smoothness

$f : \mathfrak{X} \longrightarrow \mathfrak{Y}$

$$D_{\text{ét}}(\mathfrak{X}, \Lambda) \underset{f^!}{\overset{f_!}{\rightleftarrows}} D_{\text{ét}}(\mathfrak{Y}, \Lambda)$$

adjoint functors

Definition

f is cohomologically smooth if $f^! \simeq f^* \otimes \mathbb{L}$ universally $/\mathfrak{Y}$ with $\mathbb{L}$ invertible i.e. v-locally $\simeq \underline{\Lambda}[d]$

Equivalent to :

1. $f^!\underline{\Lambda} \otimes f^*(-) \xrightarrow{\sim} f^!(-)$ (canonical morphism=iso)
2. $f^!\underline{\Lambda}$ invertible

Important property : coho. smoothness v-local / base

Examples of coho smooth morphisms

- $f : X \to Y$ morphism of "classical" rigid spaces $/K$
  f smooth $\Rightarrow f^\diamond : X^\diamond \to Y^\diamond$ coho smooth (Huber's relative Poincaré duality)
- $\mathbb{B}_S^{d,1/p^\infty} \longrightarrow S$ with $S \in \mathrm{Perf}_{\mathbb{F}_p}$
- $$f : X \longrightarrow Y$$
  morphism of loc. spatial diamonds
  K =pro-p group acts freely on X/Y
  $$g : X/\underline{K} \longrightarrow Y$$
  Then f coho smooth $\Rightarrow g$ coho smooth

  ⚠ : $X \to X/\underline{K}$ not coho smooth
  $A =$ profinite set
  $$h : \underline{A}_{\mathrm{Spa}(C,\mathscr{O}_C)} \to \mathrm{Spa}(C,\mathscr{O}_C)$$
  étale sheaves / $\underline{A}_{\mathrm{Spa}(C,\mathscr{O}_C)} = \mathscr{C}^\infty(A,\Lambda)-$modules
  $$h^! \underline{\Lambda} = \mathscr{D}(A,\Lambda)$$
  (distributions)

Examples of coho smooth morphisms

- X_S = curve $/\mathrm{Spa}(\mathbb{Q}_p)$ "parametrized by $S \in \mathrm{Perf}_{\mathbb{F}_p}$"

  $\mathscr{E}/X_S$ vector bundle >0 H.N. slopes fiberwise on S

  $BC(\mathscr{E})$ = relative H^0 of $\mathscr{E}$: $T/S \longmapsto H^0(X_T, \mathscr{E}_{|X_T})$

  Then : $BC(\mathscr{E}) \to S$ is coho smooth

  Kedlaya-Liu $\Rightarrow$ loc.$/S$ $\exists\ 0 \to \mathscr{E}' \to \mathscr{E}'' \to \mathscr{E} \to 0$ with
  - $\mathscr{E}'$ s.s. slope 0 (fiberwise $/S$)
  - $\mathscr{E}''$ s.s. slope $\frac{1}{n}$ for some $n \geq 1$ (fiberwise $/S$)

  $$0 \longrightarrow \underbrace{BC(\mathscr{E}')}_{\substack{\text{pro-étale} \\ \text{loc} \simeq \underline{\mathbb{Q}_p}^m}} \longrightarrow \underbrace{BC(\mathscr{E}'')}_{\substack{\text{loc}/S \simeq \mathring{\mathbb{B}}_S^{d,1/p^\infty} \\ \text{univ cover } p\text{-div formal group}}} \longrightarrow BC(\mathscr{E}) \longrightarrow 0$$

  $\Rightarrow BC(\mathscr{E})/S$ is coho smooth
- Idem with $BC(\mathscr{E})$ = relative H^1, if $\mathscr{E}$ has <0 H.N. slopes fiberwise $/S$

  Example : $BC(\mathscr{O}(-1)) = (\mathbb{G}_{a,S^\sharp})^\diamond/\underline{\mathbb{Q}_p}$

An application of the examples

Starting point of discussions about smoothness with Scholze.

$G/\mathbb{Q}_p$ affine alg. group then

$$[\bullet/\underline{G(\mathbb{Q}_p)}] = \text{classifying stack of pro-étale } \underline{G(\mathbb{Q}_p)}\text{-torsors}$$

is cohomologically smooth of dimension 0. If G reductive

$$[\bullet/\underline{G(\mathbb{Q}_p)}] = \mathrm{Bun}_G^{ss,c_1=0} \overset{\text{open}}{\hookrightarrow} \mathrm{Bun}_G$$

$\rightsquigarrow$ particular case of chat we want to prove (Bun_G coho smooth)

An application of the examples

- $G = \mathrm{GL}_n$. $U \subset BC(\mathscr{O}(1))^n$ open ($\neq \emptyset$) subset of surjections

$$u : \mathscr{O}^n \twoheadrightarrow \mathscr{O}(1)$$

  s.t. $\ker(u)$ has < 0 H.N. slopes. $\underline{\mathrm{GL}_n(\mathbb{Q}_p)} = \underline{\mathrm{Aut}}(\mathscr{O}^n)$ acts freely on U.
  $K \subset \mathrm{GL}_n(\mathbb{Q}_p)$ pro-p open

$$f : U/\underline{K} \longrightarrow [\bullet/\underline{\mathrm{GL}_n(\mathbb{Q}_p)}]$$

  via $U \to \bullet$ and $\underline{K} \hookrightarrow \underline{\mathrm{GL}_n(\mathbb{Q}_p)}$
  - $U/\underline{K} \to \bullet$ is coho smooth
  - f = locally trivial fibration in $U \underset{K}{\times} \mathrm{GL}_n(\mathbb{Q}_p) = \coprod_{K \backslash \mathrm{GL}_n(\mathbb{Q}_p)} U$ = coho smooth
- Any G. Fix $G \hookrightarrow \mathrm{GL}_n$, $K \subset G(\mathbb{Q}_p)$ pro-p open and consider

$$U/\underline{K} \longrightarrow [\bullet/\underline{G(\mathbb{Q}_p)}]$$

Statement of the main theorem

$S = \mathrm{Spa}(R, R^+)$ affinoid perfectoid

$$Z \to X_S$$

smooth adic space, adification of $\mathfrak{Z} \to \underbrace{\mathfrak{X}_R}_{\text{schematical curve}}$ quasi-projective

Definition

$\mathcal{M}_Z$ =a moduli space of sections of Z/X_S

$$\mathcal{M}_Z : T/S \longmapsto \left\{ f, \begin{array}{ccc} & & Z \\ & {}^{f}\nearrow & \downarrow \\ X_T & \to & X_S \end{array}, \text{ s.t. } f^* T_{Z/X_S} \text{ has } > 0 \text{ H.N. slopes fiberwise}/T \right\}$$

Theorem

$\mathcal{M}_Z \longrightarrow S$ is a cohomologically smooth morphism of locally spatial diamonds

Examples of spaces $\mathscr{M}_Z$

- *linear case* : $Z = \mathbb{V}(\mathscr{E})$, $\mathscr{E}$ = vector bundle on X_S

$$\mathscr{M}_Z = BC(\mathscr{E}) \times_S U$$

  where $BC(\mathscr{E})$ =relative H^0 of $\mathscr{E}$, U =open subset of S where $\mathscr{E}$ has >0 H.N. slopes
- *projective space case* : $Z = \mathbb{P}(\mathscr{E})$

$$\mathscr{M}_Z = \coprod_{d \in \mathbb{Z}} U_d / \underline{\mathbb{Q}_p}^\times$$

  $U_d \subset BC(\mathscr{E}^\vee(d)) \setminus \{0\}$ open subset of surjections

$$u : \mathscr{E} \twoheadrightarrow \mathscr{O}(d)$$

  s.t. $\ker u$ has H.N. slopes $< d$
- *Gromov-Witten case* : $Z = X_S \times_{Spa(\mathbb{Q}_p)} Z'$. Then $\mathscr{M}_Z$ defined over $\mathrm{Spa}(\mathbb{F}_p)$

  $=$ moduli of morphisms $f : X_S \to Z'$ s.t. $f^* T_{Z'/\mathbb{Q}_p}$ is >0

Examples of spaces $\mathscr{M}_Z$

- *Reduction of a G-bundle to a parabolic subgroup* :
  $P \subset G$ parabolic subgroup

$$\mathrm{Bun}_P^{>0} \subset \mathrm{Bun}_P$$

open substack

$$\mathrm{Bun}_P^{>0} = \{P\text{-bundles } \mathscr{E} \mid \mathscr{E} \underset{P}{\times} \mathfrak{g}/\mathfrak{p} \text{ has } >0 \text{ H.N. slopes}\}$$

$Ad : P \to \mathrm{GL}(\mathfrak{g}/\mathfrak{p})$ (adjoint representation).
Theorem implies :

$$\mathrm{Bun}_P^{>0} \longrightarrow \mathrm{Bun}_G \text{ is coho smooth}$$

$\rightsquigarrow$ nice coho smooth charts on Bun_G, $\pi_b : \mathscr{M}_b \to \mathrm{Bun}_G$ (see Scholze's IHES lectures)

Example : $G = \mathrm{GL}_n$, P =Borel, $\mathrm{Bun}_P^{>0}$ = moduli of

$$0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_n = \mathscr{E}$$

full flag s.t.

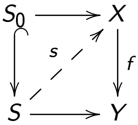
$$\deg(\mathscr{E}_i/\mathscr{E}_{i-1}) < \deg(\mathscr{E}_{i+1}/\mathscr{E}_i)$$

"anti-HN filtration"

Proof of the main theorem : formal smoothness

Definition

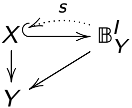
$f : X \longrightarrow Y$ is formally smooth if for any diagram



where $S_0 \hookrightarrow S$ is a Zariski closed embedding of aff perf spaces, $\exists s$ (up to replacing S by an étale neighborhood of S_0)

→ More near from the topological notion of Euclidean Neighborhood Retract (Borsuk) than Grothendieck's algebraic notion.

Example : X, Y aff perf



for some set I. Then

$$X/Y \text{ f.s. } \Leftrightarrow \exists \text{ retraction } s$$

(up to replacing $\mathbb{B}^I_Y$ by an étale neighborhood of X)

Proof of the main theorem : formal smoothness

Theorem

The morphism $\mathscr{M}_Z \longrightarrow S$ is formally smooth

Remarks :

1. "Classical" case X/k smooth projective curve, replacing ">0 H.N. slopes" in the def. of $\mathscr{M}_Z$ by "has no H^1", this is very easy (with Groth. def. of formal smoothness)
2. formally smooth does not imply coho smooth à priori. For example normal crossing divisor $\{xy=0\} \subset \mathbf{A}^2_{\mathbb{C}}$ is a topological retract but no coho smooth

Proof of the main theorem : consequence of formal smoothness

$$f : \mathcal{M}_Z \longrightarrow S$$

f formally smooth $\Rightarrow$ $\underline{\Lambda}$ is f-U.L.A. (see Scholze's talk)
$\Rightarrow f^!\underline{\Lambda} \otimes f^*(-) \xrightarrow{\sim} f^!(-)$

Thus it remains to prove that $f^!\underline{\Lambda}$ is invertible.

Use : f formally smooth $\Rightarrow$ the dualizing complex $\mathbb{D}_{\mathcal{M}_Z/S} := f^!\underline{\Lambda}$ is compatible with base change : for any $S' \to S$, if

$$\begin{array}{ccc} \mathcal{M}_Z \times_S S' = \mathcal{M}_{Z \times_{X_S} X_{S'}} & \xrightarrow{h} & \mathcal{M}_Z \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

$$h^* \mathbb{D}_{\mathcal{M}_Z/S} = \mathbb{D}_{\mathcal{M}_{Z \times_{X_S} X_{S'}}/S'}$$

$\Rightarrow$ one can suppose $\mathcal{M}_Z \to S$ has a section s and we want to prove

$$s^* \mathbb{D}_{\mathcal{M}_Z/S}$$

is invertible.

Proof of the main theorem : deformation to the normal cone

$s =$ section of $\mathscr{M}_Z/S$ corresponds to σ =section of Z/X_S

$$\sigma : X_S \hookrightarrow Z.$$

$\widetilde{Z \times \mathbb{B}^1}$ = blow-up of $Z \times \mathbb{B}^1$ along $(\sigma, 0)$ (deformation to the normal cone of $\sigma : X_S \hookrightarrow Z$)
Considering

$$(\widetilde{Z \times \mathbb{B}^1}) \times_{\mathbb{B}^1} \mathbb{B}^{1,1/p^\infty}$$

one defines ($\mathbb{B}_S := \mathbb{B}_S^{1,1/p^\infty}$)

$$\begin{array}{c} \widetilde{\mathscr{M}} \\ \widetilde{s}\uparrow \downarrow \rho \\ \mathbb{B}_S \end{array}$$

satisfying

- $\rho^{-1}(\mathbb{B}_S \setminus \{0\}) = \mathscr{M}_Z \times_S (\mathbb{B}_S \setminus \{0\})$ and $\widetilde{s} \leftrightarrow (s, Id)$
- $\widetilde{s}_{|\rho^{-1}(0)} : S \overset{\text{zero section}}{\hookrightarrow} BC(\underbrace{\sigma^* T_{Z/X_S}}_{\text{conormal bundle to } \sigma}) \overset{\text{open}}{\hookrightarrow} \rho^{-1}(0)$

Proof of the main theorem : deformation to the normal cone

$$\widetilde{\mathscr{M}} \xrightarrow{\rho} \mathbb{B}_S, \quad \widetilde{s} : \mathbb{B}_S \to \widetilde{\mathscr{M}}$$

$\varpi \in \mathscr{O}(S) =$ p.u. element

$$\times\varpi : \mathbb{B}_S^{1,1/p^\infty} \to \mathbb{B}_S^{1,1/p^\infty}$$

multiplication by ϖ.

$K := \widetilde{s}^* \rho^! \underline{\Lambda} =$ complex of $\varpi^{\mathbb{N}}$-equivariant étale sheaves on $\mathbb{B}_S$ s.t. if $i : S \hookrightarrow \mathbb{B}_S^{1,1/p^\infty}$ (zero section)

$$i^* K$$

is invertible (coho smoothness of $BC(s^* T_{Z/X_S})$ + formal smoothness of ρ)

$\implies K$ is invertible using that $\times\varpi : \mathbb{B}_S^{1,1/p^\infty} \to \mathbb{B}_S^{1,1/p^\infty}$ is contracting.

$\implies$ the result looking at $K_{|\mathbb{B}_S \setminus \{0\}}$.

A Jacobian criterion

$d > 0$, $P \in \mathbb{Q}_p[X_1, \dots, X_n]$ homogeneous of degree δ
$BC(\mathscr{O}(d)) = (B^+_{cris})^{\varphi = p^d}$

$$\widetilde{P} : BC(\mathscr{O}(d))^n \longrightarrow BC(\mathscr{O}(d\delta))$$

For $x \in BC(\mathscr{O}(d))^n(C)$

$$Jac_{\widetilde{P},x} : \mathscr{O}(d)^n \longrightarrow \mathscr{O}(d\delta)$$

linear morphism between v.b./X_C.

U = open subset where $Jac_{\widetilde{P},x}$ is surjective with > 0 kernel.

Then $U \to \mathrm{Spa}(\mathbb{F}_p)$ is cohomologically smooth.

Conclusion

Good notion = formally smooth + cohomologically smooth

Pattern : prove first something is formally smooth $\rightsquigarrow$ much easier then to prove this is coho smooth